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A Note on Decision Rules for Stochastic Programs

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In this note it will be shown that, in a sense to be made precise, a two-stage stochastic program with recourse with right-hand sides random (i.e., a two-stage programming under uncertainty problem) has optimal decision rules which are *continuous* and *piecewise linear*. The proof relies on a basic property of linear programs established in [9]. However, this result does not extend to stochastic programs with three or more stages. An example will be given of a simple inventory-type three-stage stochastic program with recourse for which the optimal second-stage decision rule is not piecewise linear. The example is then recast in the framework of the conditional probability E-model of chance-constrained programming given by Charnes and Kirby in [1], showing that the central theorem of [1], [4], and [5] on the existence of piecewise linear decision rules for such programs is invalid for more than two stages. The example may also be of value to the reader as an illustration of certain concepts in the theory of stochastic programs with recourse.

Consider the two-stage stochastic program with recourse with right-hand sides random:

$$
\begin{aligned}
\inf_x \quad & cx + E_\xi\{\min_y \; qy\} \\
& Ax \qquad\qquad\;\; = b \\
& Tx \qquad + Wy = \xi \\
& \;\; x \geqslant 0 \qquad\; y \geqslant 0
\end{aligned}
\tag{1}
$$

where c, q, A, b, T, and W are fixed matrices of size

$$1 \times n,\ 1 \times \bar{n},\ m \times n,\ m \times 1,\ \bar{m} \times n, \text{ and } \bar{m} \times \bar{n}$$

respectively and ξ is a random column $\bar{m}$-vector with known distribution. The *support* $\tilde{\Xi}$ of the random vector ξ is defined to be the smallest closed set of values of ξ with probability measure 1. This special case of stochastic programs with recourse is essentially the programming under uncertainty model which is studied in [3], [11], etc. and out of which the study of stochastic programs with recourse grew. Of course as it stands (1) is just a notational abbreviation. A detailed explanation of (1) may be found in [*11*]; a similar explicit formulation for the general stochastic program with recourse, in which the components of ξ include elements of c, q, T, and W as well as the right-hand sides, may be found in [*6*]. The pertinent concepts and results from [*6*] and [*11*] are outlined in the following paragraph.

For every pair of values of the vectors x and ξ, $Q(x, \xi)$ is defined to be the optimal value of the *second-stage program*

$$\begin{aligned} \min_y \quad & qy \\ Wy &= \xi - Tx \\ y &\geqslant 0 \end{aligned} \tag{2}$$

where it is understood that $Q(x, \xi)$ is $+\infty$ or $-\infty$ if (2) is infeasible or feasible and unbounded below. The usual definition of integration is extended in [*6*] to encompass the cases of divergence and infinite integrand. Thus $Q(x) = E_\xi Q(x, \xi)$, the expected value of $Q(x, \xi)$, can be unambiguously defined for all x. It is shown in [*8*] that with this definition of expected value and the appropriate definition of convexity for functions into the extended reals $Q(x)$ is a lower semicontinuous convex function. Thus (1) has an *equivalent convex program*

$$\begin{aligned} \inf_x \quad & cx + Q(x) \\ Ax &= b \\ x &\geqslant 0 \end{aligned} \tag{3}$$

where, of course, $Q(x)$ may be infinite for certain values of x. A vector x is said to be (strongly) *feasible* for (1), or equivalently (3), if it satisfies the linear constraints $Ax = b$, $x \geqslant 0$ and if $Q(x) < +\infty$. If the infimum in (1), i.e., in (3), is finite and achieved for some vector x, then x is said to be *optimal*.

The major positive result of this note is the following:

THEOREM. *If a two-stage stochastic program with recourse with right-hand sides random* (1) *has a feasible solution x^0 yielding a finite value of the objective, in particular if x^0 is an optimal solution, then there exists a continuous piecewise linear vector-valued function $y^0(\cdot)$ defined on a polyhedral region $\Sigma(x^0)$ containing the support $\tilde{\Xi}$ of the random variable ξ such that for all ξ in $\Sigma(x^0)$, and hence for all ξ in $\tilde{\Xi}$, $y^0(\xi)$ is a finite optimal solution to the second-stage program* (2) *given $x = x^0$.*

Proof. Let pos W denote the closed convex polyhedral cone with apex at the origin consisting of all column $\bar{m}$-vectors which can be written as a nonnegative weighted sum of columns of W, i.e., pos $W = \{t \mid t = Wy, y \geqslant 0\}$. The second-stage program (2) is feasible for given x and ξ if and only if ξ lies in the translated cone $\Sigma(x) =$ pos $W - Tx$. Since x^0 is feasible, we have $Q(x^0) = E_\xi Q(x^0, \xi) < +\infty$.

From the definition of expectation given in [6] it follows that $Q(x^0, \xi)$ must be less than $+\infty$ with probability 1, i.e., $\Sigma(x^0)$ must have measure 1. But $\Sigma(x^0)$ is closed and $\bar{\Xi}$ is the smallest closed set of measure 1, hence $\Sigma(x^0) \supset \bar{\Xi}$. Now if the second-stage program (2) is unbounded for one feasible right-hand side, it is unbounded for all feasible right-hand sides. That is, if $Q(x^0, \xi) = -\infty$ for some ξ, then $Q(x^0, \xi) = -\infty$ for all ξ in $\Sigma(x^0)$. But if $Q(x^0, \xi) = -\infty$ for all ξ in $\Sigma(x^0)$, then, by the definition of E, $Q(x^0) = E_\xi Q(x^0, \xi) = -\infty$. Since x^0 yields a finite value of the objective in (3), it follows that the second-stage program (2) has a (finite) optimal solution $y(\xi)$ for $x = x^0$ and all ξ in $\Sigma(x^0)$. Up to this point, the proof of the theorem is essentially a review of some of the results of [6] and [11]. We may now apply the Basis Decomposition Theorem of [9] to conclude that the optimal solution $y^0(\xi)$ may be chosen for each ξ in $\Sigma(x^0)$ in such a way that $y^0(\cdot)$ is continuous and piecewise linear on $\Sigma(x^0)$.

In view of this theorem, it is natural to ask if programs of more than two stages also have piecewise linear optimal decision rules. They do not in general, as the following example will show. Consider the three-stage stochastic program with recourse:

$$\begin{array}{llllll} \min\limits_x & 0.6x + E_\xi\{\min\limits_y & 0y_1 + 0y_2 + E_{(\zeta|\xi)}\{\min\limits_z & z_1 + z_2\}\} & \\ & x & & & \leqslant 2 \\ & -x & +y_1 + \ y_2 & & \leqslant \xi \\ & & y_1 & +z_1 & \geqslant \zeta_1 \\ & & \quad y_2 & \quad +z_2 & \geqslant \zeta_2 \\ & & & x, y_1, y_2, z_1, z_2 & \geqslant 0 \end{array} \tag{4}$$

where ξ, ζ_1, and ζ_2 are independent random variables, ξ is uniformly distributed on [0,2], ζ_1 is uniformly distributed on [0,1], and ζ_2 is triangularly distributed on [0,1], i.e., $P(\zeta_2 \leqslant \lambda) = \lambda^2$. This three-stage program may be interpreted as an inventory problem. An amount x of an infinitely divisible commodity may be purchased at a price of 0.6 per unit. An additional amount ξ is then received at no cost, and amounts y_1 and y_2 of the total are sent to two distribution facilities. If either of the demands ζ_1 and ζ_2 exceeds the supply at the corresponding facility, the deficiency is made up at a cost of 1 per unit. Notice that this program has the property of complete recourse as described in [3], that is, the second stage is always feasible for any choice of x satisfying the first-stage constraints and any value of ξ, and the third stage is always feasible for any feasible choice of y_1 and y_2 and any values of ζ_1 and ζ_2.

Rewriting the constraints in equality form with slack variables s_1 and s_2, the *third-stage program* is

$$\begin{array}{lllll} Q_2(y, \zeta) = \min\limits_{z,s} & z_1 + z_2 & & & \\ & z_1 & -s_1 & & = \zeta_1 - y_1 \\ & \quad z_2 & & -s_2 & = \zeta_2 - y_2 \\ & & z_1, z_2, s_1, s_2 \geqslant 0 & & \end{array}$$

This is a case of simple reccourse as considered in [10]. By the well known separability property of such problems,

$$\begin{aligned} Q_2(y) &= E_\zeta Q_2(y, \zeta) \\ &= E_{\zeta_1} Q_2'(y_1, \zeta_1) + E_{\zeta_2} Q_2''(y_2, \zeta_2), \end{aligned}$$

where

$$E_{\zeta_1} Q_2'(y_1, \zeta_1) = Q_2'(y_1) = \begin{cases} -y_1 + \frac{1}{2} & \text{if} \quad y_1 \leqslant 0 \\ \frac{1}{2}(y_1 - 1)^2 & \text{if} \quad 0 \leqslant y_1 \leqslant 1 \\ 0 & \text{if} \quad 1 \leqslant y_1 \end{cases}$$

$$E_{\zeta_2} Q_2''(y_2, \zeta_2) = Q_2''(y_2) = \begin{cases} -y_2 + \frac{2}{3} & \text{if} \quad y_2 \leqslant 0 \\ \frac{1}{3} y_2^3 - y_2 + \frac{2}{3} & \text{if} \quad 0 \leqslant y_2 \leqslant 1 \\ 0 & \text{if} \quad 1 \leqslant y_2 \end{cases} \tag{5}$$

Fig. 1. includes a plot of selected level curves for $Q_2(y_1, y_2)$.

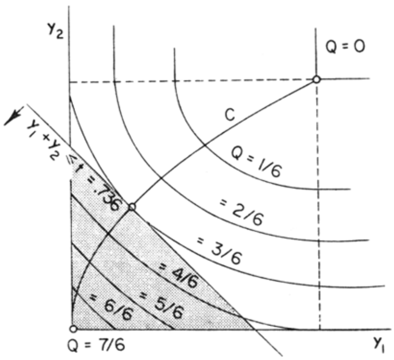

FIG. 1

The *equivalent second-stage program* is

$$\begin{aligned} Q_1(x, \xi) = \min_y \quad & 0y_1 + 0y_2 + Q_2(y_1, y_2) \\ & y_1 + y_2 \leqslant \xi + x \\ & y_1 \geqslant 0, y_2 \geqslant 0. \end{aligned} \tag{6}$$

From an examination of Figure 1 it is not difficult to see that for each value of $t = \xi + x$ in the interval [0,2] the optimal solution to (6) is unique and lies on a curve C. When $t = \xi + x$ is greater than 2 the optimal solution is not unique; if a specific solution weie desired, a natural choice would be $y_1 = y_2 = 1$. With a little straightforward analysis it can be shown that the optimal solution

$$(y_1{}^0(\xi + x), y_2{}^0(\xi + x))$$

for $0 \leqslant \xi + x \leqslant 2$ is given by

$$\begin{aligned} y_2{}^0(\xi + x) &= \frac{-1 + [1 + 4(\xi + x)]^{1/2}}{2} \\ y_1{}^0(\xi + x) &= \xi + x - y_2{}^0(\xi + x). \end{aligned} \tag{7}$$

Thus C is the parabola $y_1 = y_2{}^2$.

The *equivalent first-stage program* is

$$\min_x \quad z(x) = 0.6x + Q_1(x)$$
$$0 \leqslant x \leqslant 2$$

where

$$\begin{aligned} Q_1(x) &= E_\xi Q_1(x, \xi) \\ &= \int_x^{2+x} Q_2(y_1{}^0(t), y_2{}^0(t)) \tfrac{1}{2}\, dt \end{aligned} \tag{8}$$

As has been observed in [*3*] the objective $z(x)$ is necessarily convex; however, this can be verified directly. For values of t greater than 2 the integrand in (8) is zero. Hence the upper limit may be replaced by 2 and

$$(d/dx)\, z(x) = 0.6 - \tfrac{1}{2} Q_2(y_1{}^0(x), y_2{}^0(x)) \tag{9}$$

From Fig. 1 it can be seen that the right-hand side of (9) is increasing in x, so that $z(x)$ is indeed convex. With a little labor the integral (8) may be evaluated in closed form, but this is not essential to the solution of the program.

On substituting $x = 0$ into the right-hand side of (9) and making use of (5) and (7) it will be found that the derivative of $z(x)$ at $x = 0$ is strictly positive. It follows that $x^0 = 0$ is the unique optimal first-stage decision for the stochastic program (4). The optimal second-stage decision as a function of the observed second-stage random variable ξ is obtained by setting $x = 0$ in (7). Since the optimal second-stage decision is unique for ξ in the interval [0,2], and since the random variable ξ is distributed continuously over this interval, it follows that any piecewise linear substitute for the optimal decision rules would be nonoptimal with probability 1 and lead to a value of the objective of the stochastic program (4) strictly less than optimal.

Finally, consider the following variant of (4):

$$
\begin{array}{lllllll}
\max & E(-0.6x & & & -z_1 - z_2) & & \\
& P(b_1 \geqslant x) \geqslant 0.9 & & & & & \\
& P(b_2' \geqslant & -x + y_1 + y_2 \mid b_1) \geqslant 0.9 & & & & \\
& P(b_2'' \geqslant & -y_1 - y_2 \mid b_1) \geqslant 0.9 & & & & \\
& P(b_3' \geqslant & -y_1 & & -z_1 & \mid b_1, b_2', b_2'') \geqslant 0.9 & \\
& P(b_3'' \geqslant & & -y_2 & -z_2 & \mid b_1, b_2, b_2) \geqslant 0.9 & \\
& & & & & x, y_1, y_2, z_1, z_2 \geqslant 0 & \qquad (10)
\end{array}
$$

where

$$
\begin{array}{ll}
b_1 = \xi + 1.8 & \\
b_2' = \xi + \zeta_1 - 0.1 & b_3' = -\zeta_1 + \eta - 0.1 \\
b_2'' = \zeta_2 & b_3'' = -\zeta_2 + \eta - 0.1,
\end{array}
$$

ξ, ζ_1, and ζ_2 are independent random variables with the distributions given in connection with the recourse problem (4), η is a random variable independent of ξ, ζ_1, and ζ_2, with uniform distribution on [0,1], and $P(b_2' \geqslant \cdots \mid b_1)$ denotes conditional probability with respect to b_1, etc. It can be seen immediately that this program is a conditional probability E-model chance constrained program as defined by Charnes and Kirby in [*1*]. It is shown in [*1*] that essentially any such program is equivalent to another in which the chance constraints are replaced by constraints involving the fractile points of the conditional distribution functions of the random variables. It is easy to see[1] that the fractile form of the stochastic program with chance constraints (10) is exactly the stochastic program with recourse (4) except for an additional constraint $y_1 + y_2 \geqslant -(0.1)^{1/2}$, which arises from the second of the two second-stage constraints in (10). Since the addition of this constraint clearly does not affect the form of the optimal decision rules (7), they are also optimal decision rules for (10). This example shows that Theorem 2 of [*1*], which asserts that the optimal decision rules are piecewise linear, is invalid for more than two stages. This same theorem has also appeared in [*4*] and [*5*] and is relied upon in an essential fashion in the proof of Theorem 3 of [*2*], which considers optimal decision rules for the P-model of chance constrained programming. Note that if the random variables are discretely distributed on a *finite* set these theorems are valid but trivial, since *any* function defined on a finite subset of R^k has a piecewise linear extension to R^k.

[1] The case of two-stage programs is examined in detail in Section 5 of [*7*].

Note added in proof. A different example of a conditional probability chance-constrained program for which the optimal decision rule is not piecewise linear has recently been given by M. Eisner, R. Kaplan, and J. Soden in "Admissible Decision Rules for the E-model of Chance-Constrained Programming", Report 47, Department of Operations Research, Cornell University (June 1968).

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